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# An Output Feedback Controller Design for A Linear Spring Connected Double Inverted Pendulum: An LMI Approach

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**Abstract**

A double inverted pendulum system which is nonlinear and unstable is modified by connecting the mass carrying it with an additional spring. Also, the control input is applied to the additional mass instead of the mass carrying the pendulum. The new system is a linear spring connected double inverted pendulum as proposed by Hou et al.[1] and [2]. Therefore, according to this modification, the original double inverted pendulum system becomes more complicated and challenging problem in the nonlinear control and stabilization area. Fortunately, even though the linear spring connected double inverted pendulum is a nonlinear problem as a general double inverted pendulum, the system can be approximated and simplified as a linear time invariant system by the linearization of the equilibrium point. Therefore, many techniques in linear full state feedback control can be applied to stabilized the system such as PID control, LQR, etc. instead of using nonlinear control which is more complicated. However, in practical usage, all state variables are not measurable or in some situations the designer intends to reduce the number of measurement signals for economic reason. The purpose of this paper is to show that an LMI-based output feedback method can be used to stabilize to system when all state variables are not all available. The LMI-based output feedback method is applied to solve an example of linear spring connected double inverted pendulum and compared with LQR method. Finally, the simulation results of both methods are shown and discussed.

*Keywords:* LMI-based output feedback, a linear spring connected double inverted pendulum, optimal control, and LQR technique.

## 1. Introduction

A challenging problem, a linear spring connected double inverted pendulum as proposed in Hou et al. [1,2] and Hongxing [5], is constructed by modifying a double inverted pendulum system which is nonlinear and unstable system. The modification is done by connecting an additional mass to the mass carrying a double inverted pendulum with a linear spring. With this modification the original double inverted pendulum system become more complicated and interesting problem [5] in nonlinear control area. Many techniques in nonlinear control can be applied to stabilize the system; however, the system can be simplified by linearization at the

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equilibrium point. With the aid of linearization of the system, the simple techniques in modern control and optimal control system can be made available to solve this problem. Examples are the LQR, PID control. Furthermore, the problem can be considered as a dynamic optimization problem as proposed in [2]. In reality, the important aspect is that not all state variables are measureable, or in some situations, the designers intend to reduce the number of measurement signals for economic reasons. Therefore, some techniques such as full state feedback, LQR, etc., may not be appropriate in this situation. LMI-based output feedback is the appropriate method in order to stabilize the linearized system.

In this paper, first, a mathematical model representing the linear spring connected inverted pendulum is provided. Next, the methodology part about optimal control using the LMI-based output feedback is stated. Then, the simulation results of both LQR and the LMI-based output feedback are presented. Finally, discussions and conclusions are stated.

## 2. A Linear Spring Connected Double Inverted Pendulum System

### *2.1 Mathematical Model*

There is a linear spring connected double inverted pendulum as proposed by Hou et al.[1, 2] and Hongxing [5] as shown in figure1. The system contains 4 degree of freedoms which are angles of double inverted pendulum and displacements of carts. The mathematical model representing behavior of the system can be constructed based by using Lagrange equation or Newton's Law as shown in Eq. (1) ,Hou et al.[1, 2].

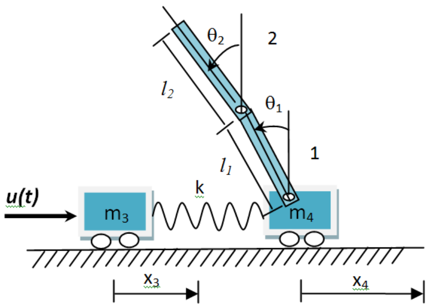


Fig. 1. A Linear spring Connected Double Inverted Pendulum System

$$-m_2 l_{C2}\cos\theta_2 \ddot{x}_4 + m_2 l_1 l_{C2}\cos(\theta_1-\theta_2)\ddot{\theta}_1 + (J_{C2}+m_2 l_{C2}^2)\ddot{\theta}_2$$
$$= m_2 g l_{C2}\sin\theta_2 + \sin\theta_2 + m_2 l_1 l_{C2}\sin(\theta_1-\theta_2)\dot{\theta}_1^2 - c_2(\dot{\theta}_1-\dot{\theta}_2)$$
$$-(m_1 l_{C1}+m_2 l_1)\cos\theta_1 \ddot{x}_4 + (J_{C1}+m_1 l_{C1}^2+m_2 l_1{}^2)\ddot{\theta}_1 + m_2 l_1 l_{C2}\cos(\theta_1-\theta_2)\ddot{\theta}_2$$
$$= m_1 g l_{C1}\sin\theta_1 + m_2 g l_1\sin\theta_1 - m_2 l_1 l_{C2}\sin(\theta_1-\theta_2)\dot{\theta}_2^2 - c_1\dot{\theta}_1 - c_2(\dot{\theta}_1-\dot{\theta}_2)$$
$$(m_4+m_1+m_2)\ddot{x}_4 - (m_1 l_{C1}+m_2 l_1)(\mu\sin\theta_1+\cos\theta_1)\ddot{\theta}_1 - (\mu m_2 l_{C2}\sin\theta_2 + m_2 l_{C2}\cos\theta_2)\ddot{\theta}_2$$
$$= \mu(m_1 l_{C1}+m_2 l_1)\dot{\theta}_1^2\cos\theta_1 + \mu m_2 l_{C2}\dot{\theta}_2^2\cos\theta_2 - (m_1 l_{C1}+m_2 l_1)\dot{\theta}_1^2\sin\theta_1 - m_2 l_{C2}\dot{\theta}_2^2\sin\theta_2$$
$$-k(x_4-x_3) - \mu(m_4+m_1+m_2)g - c\dot{x}_4$$
$$m_3\ddot{x}_3 = F(t) + k(x_4-x_3) - \mu m_3 g - c\dot{x}_3 \qquad (1)$$

where $m_3$ = mass of cart 3, $m_4$ = mass of cart 4, $c_0$ = friction factor, $c$ = tack coefficient, and $u(t)$ = control input.

### *2.2 Problem Statement*

The system can be written in the form of state space which is suitable for applying optimal control techniques. The state variables can be defined as $z_1 = x_3, z_2 = x_4, z_3 = \theta_1, z_4 = \theta_2, z_5 = \dot{x}_3, z_6 = \dot{x}_4$, $z_7 = \dot{\theta}_1$,and $z_8 = \dot{\theta}_2$. Then, the equation of motion in Eq. (1) can be converted in the form of a first order differential equation, $\underline{z} = f(\underline{z}(t), u(t))$ where $\underline{z}(t) = [z_1, z_2, z_3, z_4, z_5, z_6]^T$. After linearization, the corresponding linear system can be expressed in the state space form as

$$\dot{\underline{z}} = A\underline{z} + Bu \text{ and } y = C\underline{z} + Du \tag{2}$$

where u(t) = control input, y(t) = output variable, A = State matrix, B = input matrix, C = Output matrix and D = input-to-output coupling matrix.

## 3. LMI-Based Output Feedback Control

The LMI-Based Output Feedback is a suitable method to stabilize the linear spring connected double inverted pendulum as expressed in (1) when not all state variables are not measurable, or the number of output is reduced by the intension of the designers.

A linear matrix inequality (LMI) is regarded as a convex constraint. Consequently, optimization problems with convex objective functions and LMI constraints are solvable relatively efficiently with off-the-shelf software, i.e., MATLAB LMI Control Toolbox. The form of an LMI is very general. Linear inequalities, convex quadratic inequalities, matrix norm inequalities, and various constraints from control theory such as Lyapunov and Riccati inequalities can all be written as LMIs. Further, multiple LMIs can always be written as a single LMI of larger dimension. Thus, LMIs are a useful tool for solving a wide variety of optimization and control problems. In this paper, an output feedback controller design based on [3]-[4] can be constructed in terms of LMIs constraints.

### *3.1 Output Feedback Controller Design*

The linearized systems are derived in the previous section and designed to achieve the performance requirements given in the Problem Statement by using a full order output feedback controller. We use an LMI-based controller design methodology to achieve the following.

**Theorem** For the linearized system, the closed-loop system can accomplish the expected performance requirements: All closed-loop poles are located in the open left-half plane if and only if there exist symmetric matrices $X$, $Y$ and matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$ such that the following LMIs simultaneously satisfied.

$$\begin{pmatrix} \Omega_{11} & \Omega_{21}^T \\ \Omega_{21} & \Omega_{22} \end{pmatrix} < 0, \quad \begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0$$

where

$$\Omega_{11} = AX + XA^T + B_u\overline{C} + (B_u\overline{C})^T,\ \Omega_{21} = \overline{A} + (A + B_u\overline{D}C_y)^T,\ \Omega_{22} = A^TY + YA + \overline{B}C_y + (\overline{B}C_y)^T$$

for $X$ and $Y$ symmetric matrices, $X > Y$ means $X - Y$ is positive definite. A dynamic output feedback controller can be constructed as follows:

$$A_K := (N^T)^{-1}(\overline{A} - NB_KC_yX - YB_uC_KM^T - Y(A + B_uD_KC_y)X)(M^T)^{-1},$$

$$B_K := (N^{T)})^{-1}(\overline{B} - YB_uD_K),$$

$$C_K := (\overline{C} - D_K C_y X)(M^T)^{-1}$$

$$D_K := \overline{D}$$

where $X$ and $Y$ are arbitrary nonsingular matrices satisfying $MN^T = I - XY$ .The details of proof is purposed by Chilali and Gahinet [3].

## 4. Simulation Results and Discussion

The feasibility of the LMI-based output feedback method can be shown by simulation results of the example of linear spring connected double inverted pendulum. In the simulation, the LMI-based output feedback method is applied to solve an example of linear spring connected double inverted pendulum and compared with LQR method. The implementation is done in MATLAB, and the simulation results of both methods are shown and discussed.

The example of a linear spring connected double inverted pendulum system is defined by letting the parameters of the system as follows: $m_1 = 0.25$ kg, $m_2 = 0.25$ kg, $m_3 = 1.5$ kg, $m_4 = 1.5$ kg, $c_0 = 0$ , $c = 0$ , $l_1 = 0.4$ m, $J_{C1} = 0.0033$ kg.m$^2$, $l_{C1} = 0.2$ m, $c_1 = 0.05$ , $l_2 = 0.4$ m, $J_{C2} = 0.0033$ kg.m$^2$, $l_{C2} = 0.2$ m, $c_2 = 0.05$ , $k = 100$ N/m, $g = 9.81$ m/s$^2$ in appropriate SI units.

This example consists of two parts which represents different situations in reality. First, the displacement of both carts and the angular displacement of pendulums are measurable, thus the output signals are $y_1 = z_1 = x_3$ , $y_2 = z_2 = x_4$ , $y_3 = z_3 = \theta_1$ and $y_3 = z_4 = \theta_4$. Second, the displacements and velocities of both carts are the output signal which are $y_1 = z_1 = x_3$ , $y_2 = z_2 = x_4$ , $y_3 = z_5 = \dot{x}_3$ and $y_4 = z_6 = \dot{x}_4$ .

For the first part of the example, the simulation results given by both LMI based output feedback and LQR methods which are time response of measureable state variables, $z_1 = x_3$, $z_2 = x_4$ , $z_3 = \theta_1$ and $z_4 = \theta_4$ are shown in Fig 4 in blue curves and red dash curves respectively. The poles of the close loop control system of LMI based output feedback method located on the left half plane as shown in Fig 3. Using control input signal of the LMI-based output feedback method as shown in Fig 3 can stabilize the system.

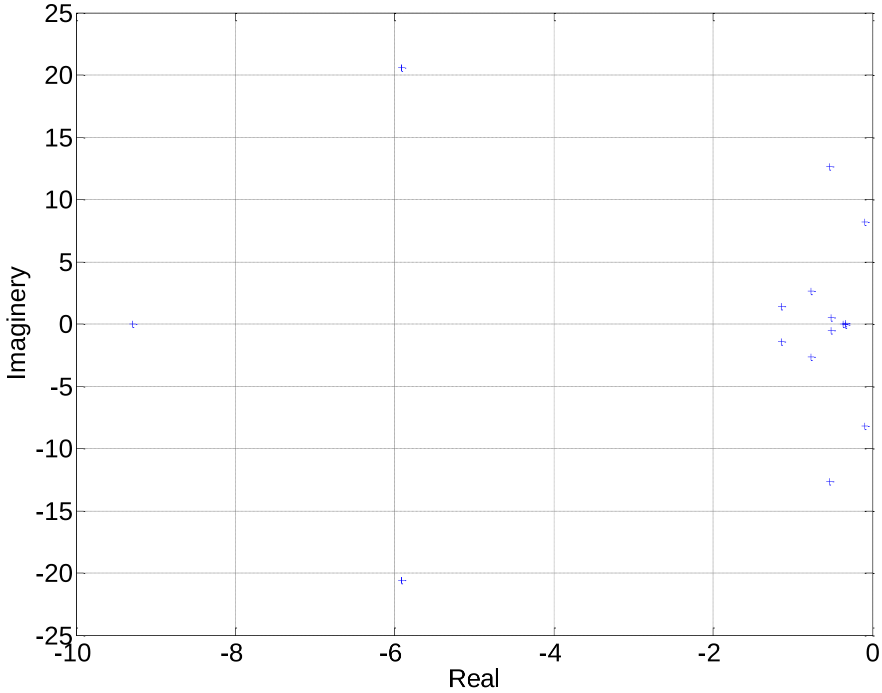


Fig. 2. Poles of the close loop control system of LMI-based output feedback method when the output signals are $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_1$ and $z_4 = \theta_2$

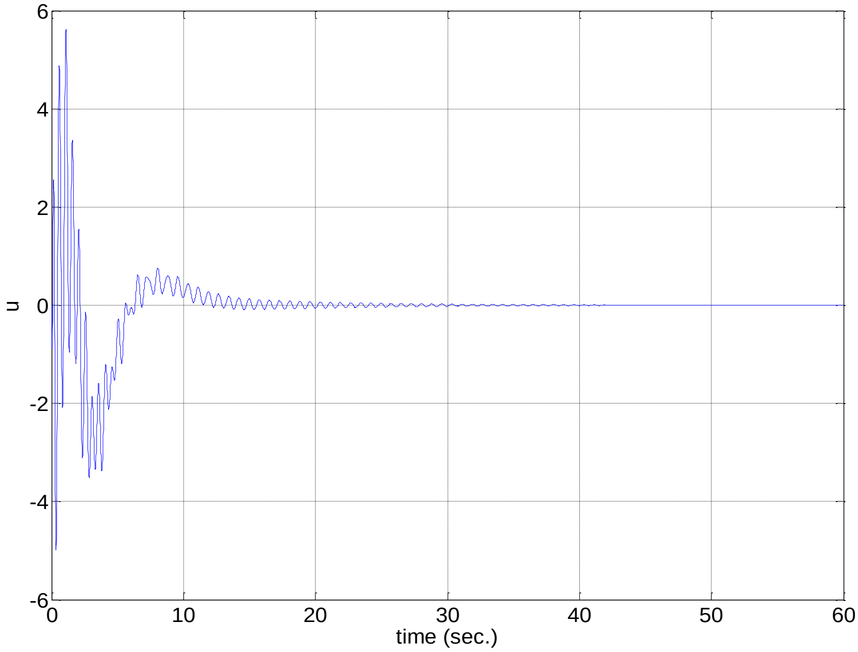


Fig. 3. A control input signal of the LMI-based output feedback method when the output signals are $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_1$ and $z_4 = \theta_2$

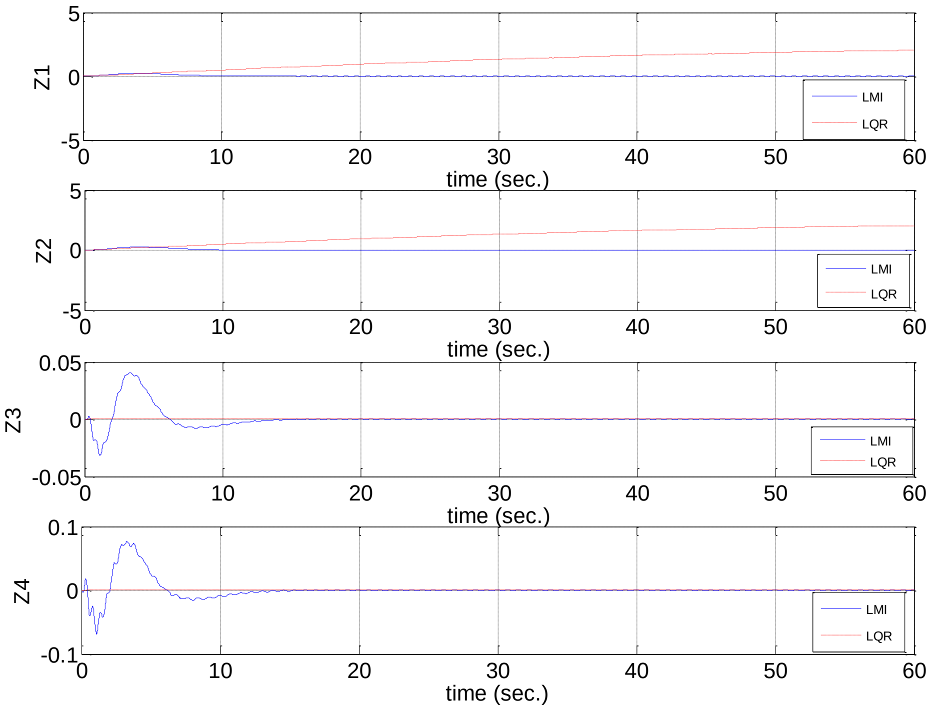


Fig. 4. Time responses of state Variables of $z_1$, $z_2$, $z_3$ and $z_4$ of the LMI-based output feedback method(blue curves) and LQR method( red dash curves) when the output signals are $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_1$ and $z_4 = \theta_2$

Next, When measurement signal are only the displacements and the velocities of the carts, the output signal are $z_1 = x_3$, $z_2 = x_4$, $z_5 = \dot{x}_3$, and $z_6 = \dot{x}_4$. The location of poles of the closed loop system of LMI-based output feedback are also on the left half plane presented in Fig 5. The control input signal of LMI-based output feedback using to stabilize the system is shown in Fig 6. The time responses of interesting state variables corresponding to displacements of the carts and the angles of the double inverted pendulum given by LMI based output feedback and LQR methods are shown in Fig 7 in blue curves and red dash curves respectively. The time responses of output signals given by LMI based output feedback and LQR methods are shown in Fig 8.

First, considering the simulation when the output signal are $y_1 = z_1$, $y_2 = z_2$, $y_3 = z_3$, $y_3 = z_4$, the LQR method can stabilized as the values of state variables $z_3 = \theta_1$ and $z_4 = \theta_2$ converge to zero . However, the carts are not at the equilibrium point at the steady state as the state variables $z_1 = x_3$ and $z_2 = x_4$ approach to finite values. Considering the simulation results given by LMI output feedback method, it is clear that LMI based output feedback method can stabilize the system as all values of state variables, $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_1$ and $z_4 = \theta_2$ converge to zero at the steady state. Therefore, it is shown that the LMI-based output feedback can stabilize the system at the equilibrium point.

Second, in the situation when output signal are $z_1 = x_3$, $z_2 = x_4$, $z_5 = \dot{x}_3$, and $z_6 = \dot{x}_4$, the LMI-based output feedback and the LQR method have the simulation results in the same way as follows. The time responses to angles of the double inverted pendulum of the system, $z_3 = \theta_1$ and $z_4 = \theta_2$ , given by LMI-based output feedback converge to equilibrium point at zero as time increases. Even though the time responses corresponding to displacements of carts, $z_3 = \theta_1$ and $z_4 = \theta_2$, do not converge to zero at steady state, both signals converge to finite value. However, considering the displacements of the carts at steady state, the time responses of the LMI-based output feedback approach closer to equilibrium point at zero than that of the LQR method. This is shown clearly in Fig 7 and Fig 8.

As simulation results of both parts of the example, it is clear that LMI based output can stabilize the system and performs better than the LQR method. Also for economically reasons, the LMI-based output method requires lower number of measurement signals than the LQR method does.

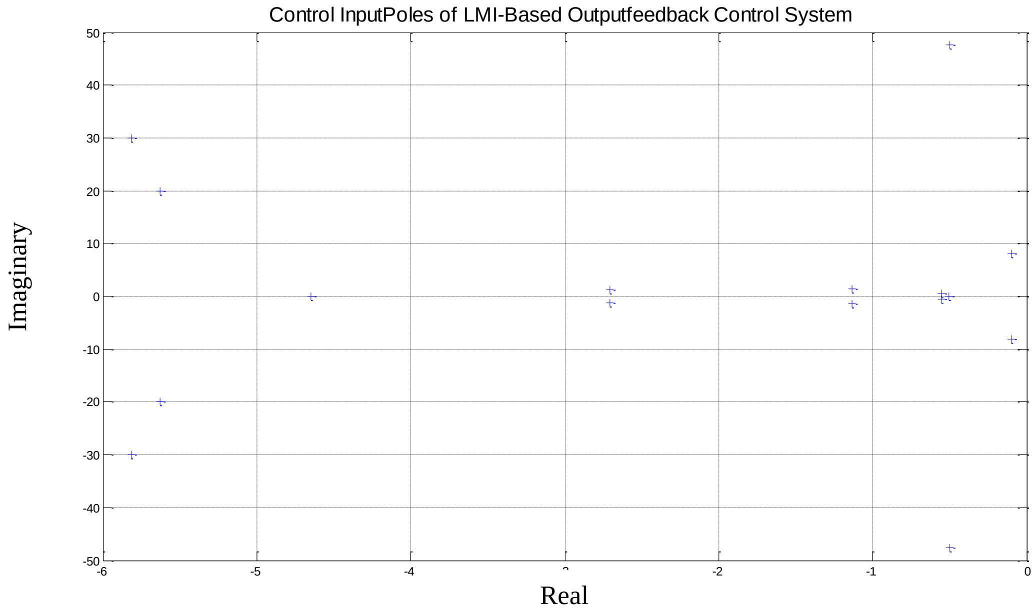


Fig. 5. Poles of the close loop control system of LMI-based output feedback method when the output signals are $z_1 = x_3$, $z_2 = x_4$, $z_5 = \dot{x}_3$, and $z_6 = \dot{x}_4$

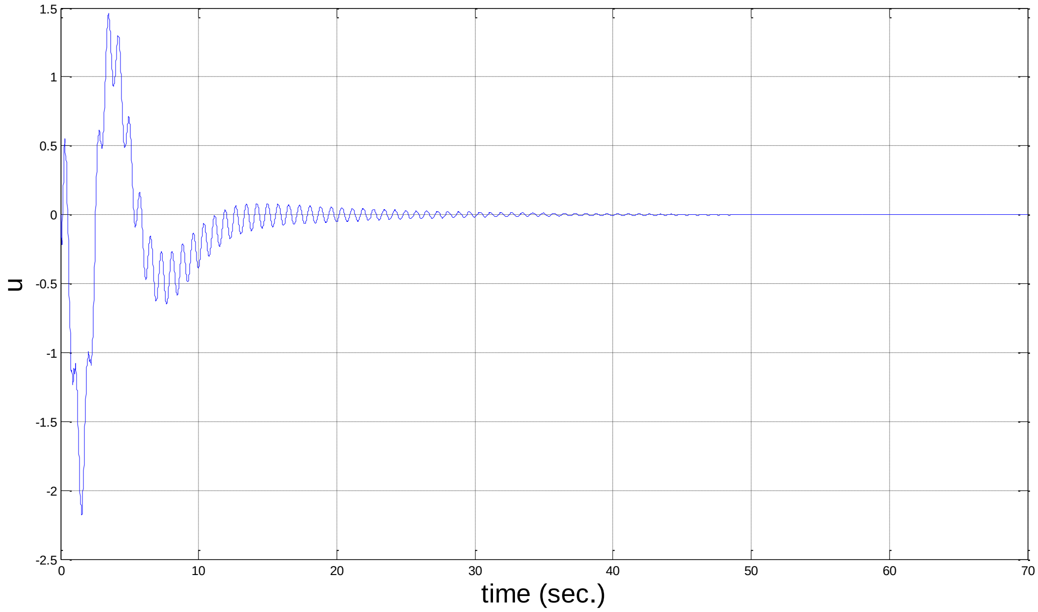


Fig. 6. A control input signal of the LMI-based output feedback method when the output signals are $z_1 = x_3$, $z_2 = x_4$, $z_5 = \dot{x}_3$, and $z_6 = \dot{x}_4$

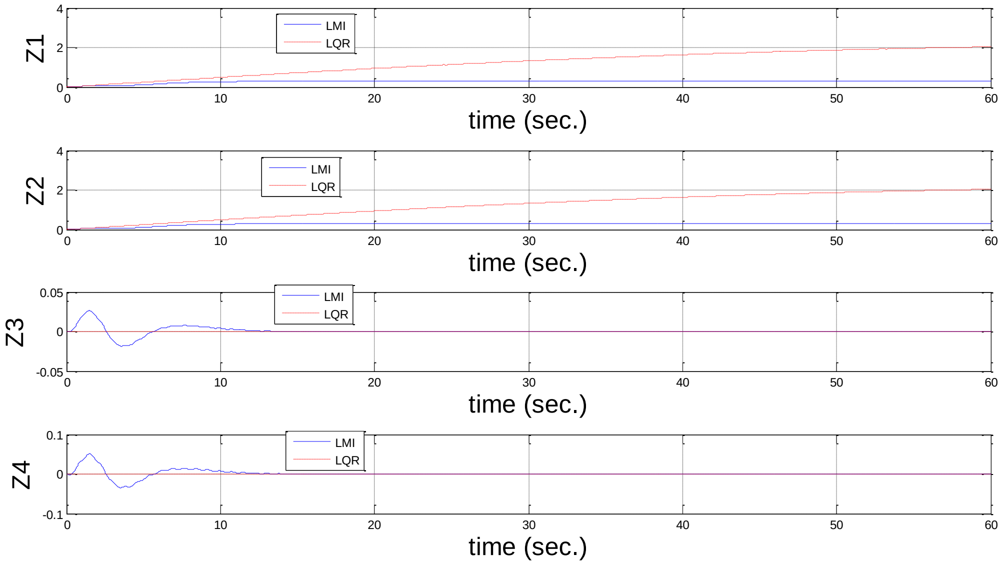


Fig. 7. Time responses of state Variables of $z_1$, $z_2$, $z_3$ and $z_4$ of the LMI-based output feedback method (blue curves) and LQR method( red dash curves) when the output signals are $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_1$ and $z_4 = \theta_2$

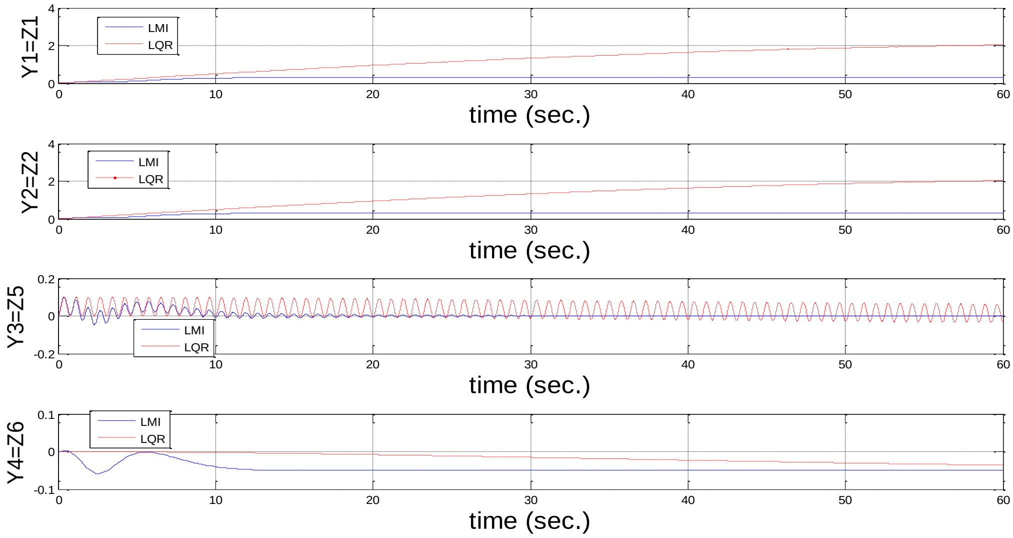


Fig. 8. Time responses of output signal of Z1 ,Z2, ,Z5 ,and Z6 of the LMI-based output feedback method (blue curves) and LQR method( red dash curves) .

## 5. Conclusions

Considering the simulation results, the following conclusions are considered:

**First,** the LMI-based output feedback is feasible to stabilize the mass spring connected double inverted pendulum system.

**Second,** the LMI-based output method performs better and more efficiently than the LQR does. The confirmations of this are as follows. The LMI-based output feedback drives all state variables converging to zero at steady state when the output signals are displacements of the carts' angles of the double inverted pendulum. Also, when the measurable signals are the displacements and velocities of the carts, at steady state both methods can stabilize the angles of the double inverted pendulum at equilibrium point, while, the displacements of the carts converge to finite value. However, the time responses of displacements of the carts given by LMI-based output feedback are closer to equilibrium point than that of the LQR method.

**Third,** the LMI-based output feedback requires lower number of measurement signals. Therefore, it is appropriate to apply an optimal control using LMI-based output feedback stabilize to a linear spring connected double inverted pendulum system for both efficiency and economical reasons.

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